LESSON

20 Integration by Parts

The integral of the product of two things is unfortunately not the product of the integrals. For example, the integral $\int x \cdot \cos(x)dx$ is not $\frac{1}{2}x^2\sin(x) + c$. We know this because the derivative of $\frac{1}{2}x^2\sin(x) + c$ is, by the Product Rule, $\frac{d}{dx}\left(\frac{1}{2}x^2\sin(x) + c\right) = 2x \cdot \sin(x) + \cos(x) \cdot \frac{1}{2}x^2$, which is not equal to $x \cdot \cos(x)$. It is unfortunate that this does not work because, if it did, evaluating integrals would be simple and would not require so many different techniques.

The integration technique that undoes the Product Rule is called *integration by parts.* The derivative of $u \cdot v$, using the Product Rule, can be expressed as $du \cdot v + dv \cdot u$ or $udv + vdu$. The corresponding integral is:

$$\int (udv + vdu) = uv$$

This can be broken up into $\int udv + \int vdu = uv$ and written as:

Integration by Parts Formula: $\int udv = uv - \int vdu$

LIPET Mnemonic

A good mnemonic for guessing what to use as u is "LIPET." That is, let u be a Logarithm if there is one. If not, let u be the Inverse of a trigonometric function (not covered in this book). If there isn't either of these, then let u be a Polynomial, and if there is none, let it be an Exponential function. Only as the very last resort, should you let u be a Trigonometric function.

This can often be used to transform a difficult integral into one that is solvable. For example, take $\int x \cdot \cos(x)dx$. This looks just like $\int u\,dv$ if $u = x$ and $dv = \cos(x)\,dx$. In order to use the formula, we will need to get du by differentiating u. Because $u = x$, we know that $\frac{du}{dx} = 1$, so $du = dx$. We will also need to get v from dv by integrating. And because $dv = \cos(x)\,dx$, it must be that $v = \sin(x)$. Thus:

$$\int x \cdot \cos(x)dx = \int u\,dv$$

$$\int x \cdot \cos(x)dx = uv - \int v\,du$$

$$\int x \cdot \cos(x)dx = x\sin(x) - \int \sin(x)\,dx$$

$$\int x \cdot \cos(x)dx = x\sin(x) + \cos(x) + c$$

This is the correct answer, as can be verified by taking the derivative $\frac{d}{dx}(x\sin(x) + \cos(x) + c) = 1 \cdot \sin(x) + \cos(x) \cdot x - \sin(x) + 0 = x \cdot \cos(x)$.

Example

Evaluate $\int xe^x\,dx$.

Solution

This cannot be solved by basic integration or by substitution, so we try integration by parts. No logarithms or inverse trigonometric functions are found here, but there is the polynomial x, so we try $u = x$. The dv must then be everything else after the integral sign, so $dv = e^x\,dx$. After differentiating u and integrating dv, we get:

$$u = x$$

$$du = dx$$

And:

$$dv = e^x\,dx$$

$$v = e^x$$

Thus, using the integration by parts formula $\int u\,dv = uv - \int v\,du$, we evaluate as follows:

$$\int xe^x\,dx = \int u\,dv$$

$$\int xe^x\,dx = uv - \int v\,du$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

$$\int xe^x\,dx = xe^x - e^x + c$$

Example

Evaluate $\int x^3 \ln(x)\,dx$.

Solution

Here, we have a logarithm, so we set $u = \ln(x)$ and $dv = x^3\,dx$. Thus:

$$u = \ln(x)$$

$$du = \frac{1}{x}dx$$

And:

$$dv = x^3\,dx$$

$$v = \frac{1}{4}x^4$$

And then we evaluate.

$$\int x^3\ln(x)\,dx = \int u\,dv$$

$$\int x^3\ln(x)\,dx = uv - \int v\,du$$

$$\int x^3\ln(x)\,dx = \ln(x)\cdot\frac{1}{4}x^4 - \int \frac{1}{4}x^4\cdot\frac{1}{x}dx$$

$$\int x^3\ln(x)\,dx = \frac{1}{4}x^4\ln(x) - \int \frac{1}{4}x^3\,dx$$

$$\int x^3\ln(x)\,dx = \frac{1}{4}x^4\ln(x) - \frac{1}{16}x^4 + c$$

This can even solve the following problem that was mentioned in Lesson 18.

Example

Evaluate $\int \ln(x)\,dx$.

Solution

Because there seems to be only one part to this integral, one wouldn't think to try integration by parts first. However, because nothing else will work, we can try $u = \ln(x)$. The only thing left for the dv is dx, so we use $dv = dx$, which leads to $v = x$.

$$u = \ln(x)$$

$$du = \frac{1}{x}dx$$

And:

$$dv = dx$$

$$v = x$$

And now evaluate as follows.

$$\int \ln(x)\,dx = \int u\,dv$$

$$\int \ln(x)\,dx = uv - \int v\,du$$

$$\int \ln(x)\,dx = \ln(x)\cdot x - \int x\cdot\frac{1}{x}dx$$

$$\int \ln(x)\,dx = x\ln(x) - \int 1\,dx$$

$$\int \ln(x)\,dx = x\ln(x) - x + c$$

Sometimes, integration by parts needs to be done more than once to solve a problem.

Example

Evaluate $\int x^2\cos(x)\,dx$.

Solution

Here, $u = x^2$, so $du = 2x\,dx$, and $dv = \cos(x)\,dx$, so $v = \sin(x)$.

$$\int x^2\cos(x)\,dx = \int u\,dv$$

$$\int x^2\cos(x)\,dx = uv - \int v\,du$$

$$\int x^2\cos(x)\,dx = x^2\sin(x) - \int \sin(x)\cdot 2x\,dx$$

$$\int x^2\cos(x)\,dx = x^2\sin(x) - \int 2x\sin(x)\,dx$$

In order to solve this $\int 2x\sin(x)\,dx$, we have to use integration by parts a second time, but this time, with $u = 2x$ and $dv = \sin(x)\,dx$.

$u = 2x$

$du = 2\,dx$

And:

$dv = \sin(x)\,dx$

$v = -\cos(x)$

Now we evaluate as follows.

$$\int x^2\cos(x)\,dx = x^2\sin(x) - \int 2x\sin(x)\,dx$$

$$\int x^2\cos(x)\,dx = x^2\sin(x) - \int u\,dv$$

$$\int x^2\cos(x)\,dx = x^2\sin(x) - (uv - \int v\,du)$$

$$\int x^2\cos(x)\,dx =$$

$$x^2\sin(x) - (2x\cdot(-\cos(x)) - \int(-\cos(x))\cdot 2\,dx)$$

$$\int x^2\cos(x)\,dx =$$

$$x^2\sin(x) + 2x\cos(x) + \int -2\cos(x)\,dx$$

$$\int x^2\cos(x)\,dx =$$

$$x^2\sin(x) + 2x\cos(x) - 2\sin(x) + c$$

The final example utilizes a clever trick that few people have ever figured out on their own. Instead, they have seen it done and learned to copy it. Opportunities to use this trick are few, but it is interesting enough to see at least once.

Example

Evaluate $\int e^x\sin(x)\,dx$.

Solution

The first letter of LIPET that we reach is E because there are neither logarithms nor polynomials, so let $u = e^x$ and $dv = \sin(x)\,dx$.

$u = e^x$

$du = e^x dx$

And:

$dv = \sin(x)\,dx$

$v = -\cos(x)$

And now evaluate as follows.

$$\int e^x \sin(x)\,dx = \int u\,dv$$

$$\int e^x \sin(x)\,dx = uv - \int v\,du$$

$$\int e^x \sin(x)\,dx =$$

$$e^x(-\cos(x)) - \int (-\cos(x)) \cdot e^x\,dx$$

$$\int e^x \sin(x)\,dx = -e^x\cos(x) + \int e^x\cos(x)\,dx$$

To evaluate $\int e^x\cos(x)\,dx$, we use integration by parts again:

$u = e^x$

$du = e^x dx$

And:

$dv = \cos(x)\,dx$

$v = \sin(x)$

And then the evaluation:

$$\int e^x \sin(x)\,dx = -e^x\cos(x) + \int e^x\cos(x)\,dx$$

$$\int e^x \sin(x)\,dx = -e^x\cos(x) + \int u\,dv$$

$$\int e^x \sin(x)\,dx = -e^x\cos(x) + uv - \int v\,du$$

$$\int e^x \sin(x)\,dx =$$

$$-e^x\cos(x) + e^x\sin(x) - \int \sin(x) \cdot e^x\,dx$$

$$\int e^x \sin(x)\,dx =$$

$$-e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx$$

Here is the moment of despair: To evaluate $\int e^x\sin(x)\,dx$, we need to be able to evaluate $\int e^x\sin(x)\,dx$! And yet, the trick here is to bring both integrals to one side of the equation:

$$\int e^x \sin(x)\,dx + \int e^x\sin(x)\,dx =$$
$$-e^x\cos(x) + e^x\sin(x)$$

$$2\int e^x \sin(x)\,dx = -e^x\cos(x) + e^x\sin(x)$$

$$\int e^x \sin(x)\,dx = \frac{1}{2}(-e^x\cos(x) + e^x\sin(x)) + c$$

▶ Practice

Evaluate the following integrals using integration by parts, substitution, or basic integration.

1. $\int x^5 \ln(x)\,dx$

2. $\int x\sin(x)\,dx$

3. $\int x\sin(x^2)\,dx$

4. $\int (x + 3)\cos(x)\,dx$

5. $\int \frac{\ln(x)}{x}\,dx$

6. $\int x^2\sin(x)\,dx$

7. $\int (x^2 + \sin(x))\,dx$

8. $\int x^2 e^{x^3+1}\,dx$

9. $\int x^2 e^x\,dx$

10. $\int (x^3 + 3x - 1)\ln(x)\,dx$

11. $\int (x + \ln(x))\,dx$

12. $\int \sqrt{x - 1}\,dx$

13. $\int x\sqrt{x - 1}\,dx$

14. $\int xe^{-x}\,dx$

15. $\int \sqrt{x}\ln(x)\,dx$

16. $\int \frac{\ln(x)}{x^3}\,dx$

17. $\int \frac{1}{x}\,dx$

18. $\int (x^2 - 1)\cos(x)\,dx$

19. $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$

20. $\int \sin(x)\sqrt{\cos(x)}\,dx$

21. $\int \sin(x) \cdot \ln(\cos(x))\,dx$

22. $\int e^x\cos(x)\,dx$